LESSON 30

Set Theory

Day 74

p. 260

Sets, Subsets, and Sample Spaces

UNDERSTAND A **set** is a collection of elements, such as objects or numbers. Imagine that you have a bag containing four cards with a number printed on each card. The numbers on the cards are listed below.

4 7 8 9

You can indicate a set by listing its elements in braces. For example, the set of numbers on the cards is {4, 7, 8, 9}, which we can refer to as set A. So, $A = \{4, 7, 8, 9\}$.

If every element in a set also belongs to another set, then the first set is a **subset** of the second set. For example, if set $B = \{4, 7\}$, then set B is a subset of set A because every element of set B is also an element of set A.

A set containing all possible elements is called the **universal set**, or parent set, denoted by the letter U. For sets A and B, the universal set might be the digits 0–9. A set containing no elements is called the empty set, or null set, and is indicated by the symbol $\varnothing$. The empty set is a subset of every set.

empty set = $\varnothing$ = { }

A set can have a **complement**, which includes all of the elements in the universal set that are not included in that set. The complement of subset B is denoted as $\overline{B}$, $\sim B$, B^c, or B'.

$\overline{B} = \{0, 1, 2, 3, 5, 6, 8, 9\}$

odd #'s: 1, 3, 5, 7, 9
even #'s: 2, 4, 6, 8.
prime #'s: 2, 3, 5, 7, 11, 13, 17, 19, 23
composite #'s: 4, 6, 9, 10, 12, 14, 15

UNDERSTAND Consider the following sets:

$C = \{7, 8\}$ $D = \{7, 11\}$

The **intersection** of sets, shown by the symbol $\cap$, consists only of the elements that the two sets have in common. You can think of the symbol $\cap$ as meaning "and." The intersection of C and D contains elements that are in set C and also in set D.

$C \cap D = \{7\}$ "overlap"

The **union** of sets, shown by the symbol $\cup$, consists of all the elements contained in either or both sets. You can think of the symbol $\cup$ as meaning "or." The union of C and D contains elements that are either in set C or in set D or in both sets.

$C \cup D = \{7, 8, 11\}$

everything in both

UNDERSTAND You can use what you know about sets to understand and describe **probabilities**. The **sample space** for a probability experiment is the set of all the possible **outcomes** for the experiment. So, the sample space for tossing a standard number cube is {1, 2, 3, 4, 5, 6}. An **event** in a probability experiment is a subset of the sample space. When a number cube is tossed, you can define any number of events, such as tossing a 2—in which case, the subset is {2}—or tossing an even number—in which case, the subset is {2, 4, 6}.

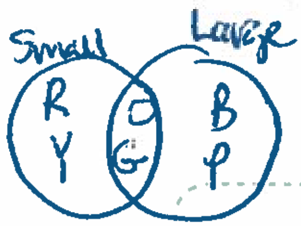
Connect

A large bag holds marbles of the following colors: blue (B), green (G), orange (O), and pink (P). A small bag holds marbles of the following colors: red (R), green (G), yellow (Y), and orange (O). Define the universal set of marble colors. Define a subset of the universal set. Find the complement of the set of colors in the small bag. Find the union and intersection of the sets of colors in the two bags.

1 Define the universal set.

The universal set is the set of all marble colors.

$U = \{B, G, O, P, R, Y\}$



2 Define a subset of the universal set.

You can define many subsets of the universal set. For now, let the colors in the large bag be set L.

$L = \{B, G, O, P\}$

Set L is a subset of the universal set U.

$\overline{L} = \{R, Y\}$

↑ Complement L', $\sim L$, L^c

3 Find the complement of the set of colors in the small bag.

Let the colors in the small bag be set S.

$S = \{R, G, Y, O\}$.

The complement of set S includes the colors that appear in the universal set (set U) but not in set S.

$\overline{S} = \{B, P\}$

4 Find the union of sets L and S.

The union of the sets contains the elements in set L or set S, or both.

$L \cup S = \{B, G, O, P, R, Y\}$

In this case, $L \cup S = U$.

$S \cup M = \{B, P, O, G, R, Y\}$

$S = \{O, G, R, Y\}$

5 Find the intersection of the sets.

The intersection of the sets contains only the elements that appear in both set L and set S.

In this case, $L \cap S = \{G, O\}$.

TRY

If $M = \{B, P\}$, what is $S \cap M$? $\varnothing$

nothing in common

Venn Diagrams

UNDERSTAND Suppose a universal set consists of all the single-digit whole numbers.

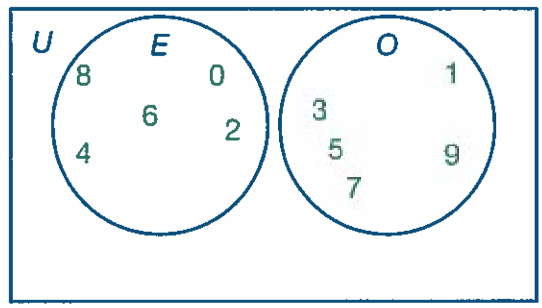
$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Many subsets can be selected from this set. For example, the set of all even numbers, set E, and the set of all odd numbers, O, would be:

$E = \{0, 2, 4, 6, 8\}$ $\quad\quad$ $O = \{1, 3, 5, 7, 9\}$

Another way to represent these sets would be to use a Venn diagram. In the Venn diagram below, the large rectangle represents the universal set, U. Subsets E and O are represented as circles inside the larger rectangle. The circles do not overlap. This means that while all of the numbers are part of the universal set, none of the numbers are common to both subsets. This is because no number is both even and odd. From observing the diagram, you can see the following:

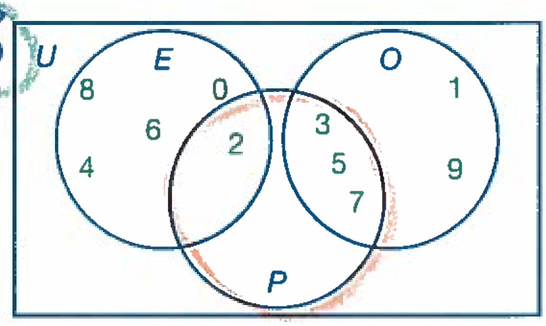
- $E \cup O = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} = U$
- $E \cap O = \varnothing$
- $\overline{E} = \{1, 3, 5, 7, 9\} = O$
- $\overline{O} = \{0, 2, 4, 6, 8\} = E$



Consider the subset of single-digit whole numbers that are also prime numbers. This set could be represented as $P = \{2, 3, 5, 7\}$. If a circle for set P is added to the Venn diagram, it will overlap with the circle for E (since 2 is prime) and with the circle for O (since 3, 5, and 7 are prime). This diagram allows us to visualize the following:

- $E \cap P = \{2\}$
- $O \cap P = \{3, 5, 7\}$
- $\overline{P} = \{0, 1, 4, 6, 8, 9\}$
- $E - P = E \cap \overline{P} = \{0, 4, 6, 8\}$

$E = \{0, 2, 4, 6, 8\}$



As shown above, there is often more than one way to describe data by using set notation. For example, to represent the values that are in set E but not in set P, use a set difference, $E - P$, or an intersection with a complement, $E \cap \overline{P}$.

It is important to consider what Venn diagrams do and do not show. For example, $\overline{P}$, the complement of set P, shows the single-digit numbers that are not prime, but that does not mean that it shows only the composite numbers. (0 and 1 are neither prime nor composite.) However, it can be used to show that 2 is the only even prime number, especially if additional whole numbers are added.

Connect

Three student journalists predicted which intramural soccer teams would make it to the semifinals of a regional tournament.

Make a Venn diagram to represent these sets of predictions.

Xavier	Yuriko	Zachary
Dolphins	Eagles	Dolphins
Eagles	Panthers	Eagles
Hawks	Sharks	Foxes
Wolves	Tigers	Tigers

1 Identify the universal set and the subsets.

The universal set will be every team in the league. This includes every team in the table, and possibly others.

The subsets are set X (Xavier's choices), set Y (Yuriko's choices), and set Z (Zachary's choices).

X = {Dolphins, Eagles, Hawks, Wolves}

Y = {Eagles, Panthers, Sharks, Tigers}

Z = {Dolphins, Eagles, Foxes, Tigers}

2 Plan the Venn diagram.

Before drawing the circles, decide if and how they will intersect.

$X \cap Y$ = {Eagles}

$Y \cap Z$ = {Eagles, Tigers}

$X \cap Z$ = {Dolphins, Eagles}

$X \cap Y \cap Z$ = {Eagles}

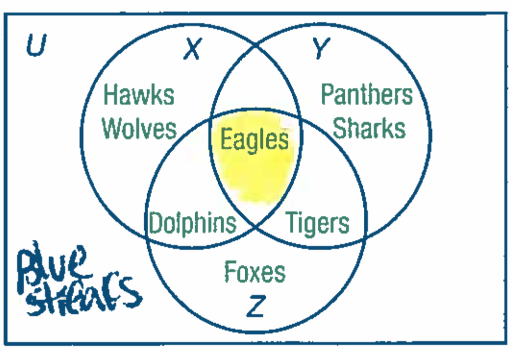
Each set overlaps the others and one element is in all three sets.

3 Fill in the Venn diagram.

Draw a rectangle to represent the universal set. Place $X \cap Y \cap Z$ (Eagles) first.

Then, place the other elements from the intersections (Dolphins, Tigers).

Then, place the remaining elements from each set.



Any teams that were not in X, Y, or Z would be written in the rectangle, but outside the circles.

MODEL

What does the empty region in the Venn diagram represent?

EXAMPLE A The table shows all of the boys in Homeroom 201. Consider this to be the universal set.

U =

~~Amal~~	~~Benji~~	~~Cheng~~	Dugan	Eric	~~Frank~~
~~Gus~~	Hal	~~Jorge~~	~~Karl~~	Leo	~~Max~~

Amal, Cheng, Frank, Gus, Jorge, and Karl are all in the art club. Consider this set A.

Benji, Cheng, Jorge, Karl, and Max are all in the environmental club. Consider this set E.

Determine $\overline{A \cup E}$, the complement of the union of sets A and E. What does that set show about the students' involvement in clubs?

1 List the universal set.

U = {Amal, Benji, Cheng, Dugan, Eric, Frank, Gus, Hal, Jorge, Karl, Leo, Max}

1st Find A∪E

2 Find the union of sets A and E.

A = {Amal, Cheng, Frank, Gus, Jorge, Karl}

E = {Benji, Cheng, Jorge, Karl, Max}

The union of the sets includes every boy in Homeroom 201 who is in either club.

So, $A \cup E$ = {Amal, Benji, Cheng, Frank, Gus, Jorge, Karl, Max}

3 Find the complement of $A \cup E$.

The complement includes all of the elements of set U that are not in the union of sets A and E. So, it shows which students are not in either club.

List the elements of the universal set and cross out elements of $A \cup E$.

{~~Amal~~, ~~Benji~~, ~~Cheng~~, Dugan, Eric, ~~Frank~~, ~~Gus~~, Hal, ~~Jorge~~, ~~Karl~~, Leo, ~~Max~~}

▶ $\overline{A \cup E}$ = {Dugan, Eric, Hal, Leo}, so, there are 4 boys in Homeroom 201 who are not in the art club or the environmental club.

who's in art AND environmental club in common

(1) $A \cap E$ = {Cheng, Jorge, Karl}

$\overline{A \cap E}$ = {Amal, Benji, Dugan, Eric, Gus, Hal, Leo, Max, Frank}

TRY

Find $\overline{A \cap E}$. Compare the number of boys in the complement of the intersection of the sets to the number of boys in the complement of the union of the sets.

EXAMPLE B Julia and Theo are playing a game in which they toss a six-sided number cube. If the cube lands on an even number, Julia gets a point. If the cube lands on a number less than 4, Theo gets a point. Create a Venn diagram to represent all the possible outcomes of a toss and who, if anyone, will get a point if it lands on that number.

1

List the universal set, or all the possible outcomes of tossing the cube.

$U = \{1, 2, 3, 4, 5, 6\}$

2

Find the outcomes that result in a point for Julia or a point for Theo.

Julia earns a point if an even number is tossed.

$J = \{2, 4, 6\}$

Theo earns a point if a number less than 4 is tossed.

$T = \{1, 2, 3\}$

The intersection of the sets shows the numbers for which both Julia and Theo would earn a point.

$J \cap T = \{2\}$

3

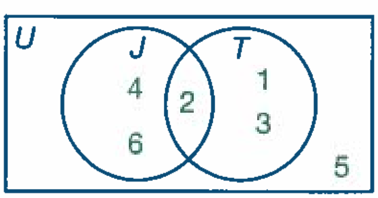
Draw the Venn diagram.

There is an intersection between the sets, so draw a rectangle with two intersecting circles inside.

Write 2 in the area where the circles overlap to show the intersection of the sets.

In the non-overlapping part of the circle for set J, write the remaining numbers from set J. Do the same for set T.

The number 5 does not earn a point for either player, so it is not written inside the circles. Since it is still part of the universal set, write it inside the rectangle but outside the circles.



DISCUSS

Shade the Venn diagram to show $J \cap T$. Then shade the diagram in another color to show $J \cup T$. How are those shadings alike? How are they different?