LESSON 34

Conditional Probability

UNDERSTAND A **conditional probability** is the probability that an event will occur given that one or more events have occurred. You can write the conditional probability of event *A* happening assuming event *B* has occurred as $P(A \mid B)$. This is read as the probability of "*A* given *B*." The conditional probability $P(A \mid B)$ is equal to the joint probability for *A* and *B* divided by the marginal probability of *B*.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

UNDERSTAND Conditional probability is not the same thing as compound probability. Suppose you toss two coins—two independent events. The sample space is {HH, HT, TH, TT}. The compound probability that both coins will land on heads is:

$$P\text{ (heads first} \cap \text{heads second)} = P(\text{heads first}) \cdot P(\text{heads second}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Suppose you already tossed the first coin and you already know that it landed on heads. The probability of both coins landing on heads is now different. Given that the first coin toss resulted in heads, there are only two possible outcomes now, {HH, HT}. So, the probability of getting {HH} is now $\frac{1}{2}$.

$$P(\text{heads second} \mid \text{heads first}) = \frac{P(\text{heads second} \cap \text{heads first})}{P(\text{heads first})} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Notice that the probability of $P(\text{heads second} \mid \text{heads first})$ is the same as $P(\text{heads second})$. If events *A* and *B* are independent, then $P(A \text{ given } B) = P(A)$ and $P(B \text{ given } A) = P(B)$. Remember that for independent events $P(A \cap B) = P(B \cap A) = P(A)P(B)$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A) \qquad P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)P(A)}{P(A)} = P(B)$$

UNDERSTAND You can use conditional probability to find a compound probability.

Multiplication Rule: The probability of the intersection of two subsets is equal to the product of the probability of one event and the conditional probability of the other event given the first event.

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A) = P(B) \cdot P(A \mid B)$$

Imagine drawing two marbles without replacement from a bag with 5 red marbles and 4 blue marbles. The probability of drawing a red marble on the first draw is $P(\text{red first}) = \frac{5}{9}$. The probability of drawing a second red marble, after the first was drawn, is $P(\text{red second} \mid \text{red first}) = \frac{4}{8} = \frac{1}{2}$. Use the Multiplication Rule to find the probability of drawing a red marble on both draws.

$$P(\text{red first and red second}) = P(\text{red first}) \cdot P(\text{red second} \mid \text{red first}) = \frac{5}{9} \cdot \frac{1}{2} = \frac{5}{18}$$

Connect

Students responding to a poll were asked whether they were for or against or had no opinion about a proposal to increase funding for the school's football program.

What is the probability that a randomly selected student at the school would be for the proposal given that the student was a girl?

	For (*F*)	Against (*A*)	No Opinion (*N*)	Total
Boys (*B*)	40	5	10	55
Girls (*G*)	10	30	5	45
Total	50	35	15	100

1 Determine which parts of the table must be considered.

The question asks for $P(F \mid G)$. It is given that the student is a girl, so look in the Girls (G) row. Find the cell that represents For (F) in that row.

2 Determine $P(F \mid G)$.

$$P(F \mid G) = \frac{P(F \cap G)}{P(G)}$$

$$= \frac{\frac{10}{100}}{\frac{45}{100}}$$

$$= \frac{10}{100} \cdot \frac{100}{45}$$

$$= \frac{10}{45} = \frac{2}{9} = 0.222\ldots$$

▶ Given that the student randomly selected is a girl, the probability that she is for the proposal is about 22%.

What is the probability that a randomly selected student at the school would be a boy, given that the student was for the proposal?

1 Determine which parts of the table must be considered.

The condition is that the person selected was for the proposal. So, we only need to consider the For (F) column. The total in that column is 50. Find the cell that represents Boys (B) in that column. The value in that cell is 40.

2 Determine $P(B \mid F)$.

The marginal frequency shows that 50 people are for the proposal. 40 of those people are boys, so

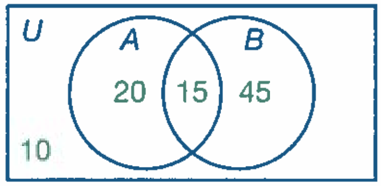
$P(B \mid F) = \frac{40}{50} = 0.8$

▶ Given that the student randomly selected is for the proposal, the probability that the student is a boy is 80%.

TRY

What is the probability that a randomly selected student at the school would have no opinion about the proposal, given that the student was a boy?

EXAMPLE A This Venn diagram shows the intersection of two sets. Set A shows the number of 10th-grade students surveyed, and set B shows the number of high school students surveyed who have computers at home. Show which parts of the Venn diagram can be used to calculate $P(A \mid B)$. Then find that probability and explain what it represents in the problem.

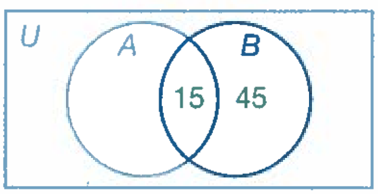


1 Determine which parts of the table must be considered.

$P(A \mid B)$ means the probability of A given B.

Since B is given, only values within the circle for B may be considered. Regions outside B must be ignored. The region "A given B" refers to the parts of region A that are also part of region B, in other words, the overlap or intersection.

The Venn diagram below shows which parts must be considered.



2 Determine $P(A \mid B)$ and explain what it represents.

Divide the joint frequency for A and B by the marginal probability of B.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{15}{15 + 45}$$

$$= \frac{15}{60}$$

$$= \frac{1}{4}$$

▶ Given that a student selected at random has a computer at home, the probability that the student is in 10th grade is $\frac{1}{4}$.

DISCUSS

Is $P(B \mid A) = P(A \mid B)$ for this situation? Why or why not?

EXAMPLE B Brenna asked 100 shoppers at a clothing store if they bought a coat that day and if they bought a shirt that day. The results of the survey are shown on the right.

Are the events buying a coat and buying a shirt independent events?

	Bought a Coat	No Coat
Bought a Shirt	3	12
No Shirt	17	68

1 Make a plan.

Let C = bought a coat.

Let S = bought a shirt.

If $P(C \mid S) = P(C)$, then C and S are independent events.

Find the marginal frequencies for the table by extending it, as shown below.

	Bought a Coat	No Coat	Total
Bought a Shirt	3	12	15
No Shirt	17	68	85
Total	20	80	100

2 Find $P(C \mid S)$.

Find the probability of buying a coat and a shirt.

$P(C \cap S) = \frac{3}{100} = 0.03$

Find the probability of buying a shirt.

$P(S) = \frac{15}{100} = 0.15$

Find the probability of buying a coat, given that you bought a coat.

$$P(C \mid S) = \frac{P(C \cap S)}{P(S)}$$
$$= \frac{0.03}{0.15}$$
$$= 0.2$$

3 Find $P(C)$ and compare.

Find the probability of buying a coat.

$P(C) = \frac{20}{100} = 0.2$

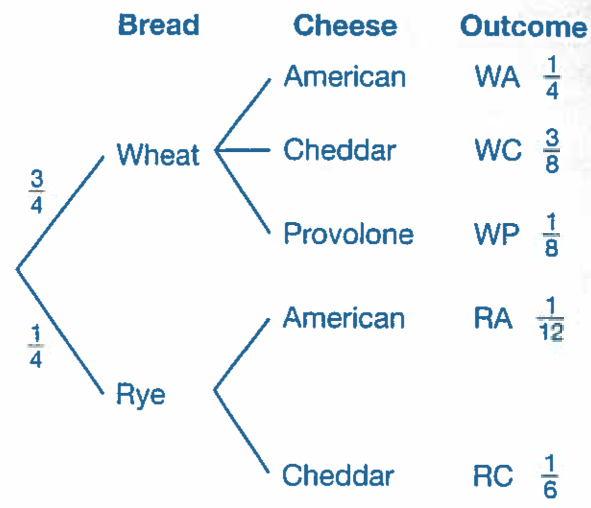
▶ $P(C \mid S) = 0.2$ and $P(C) = 0.2$ also. So, buying a coat and buying a shirt are independent events.

DISCUSS

Could you also show that events C and S are independent by proving that $P(S \mid C) = P(S)$? Explain and show your work.

EXAMPLE C Antoine made 24 cheese sandwiches for a family picnic. For each sandwich, he used one kind of bread and one kind of cheese. The tree diagram shows the probability of choosing each sandwich combination.

Suppose Antoine's cousin is the first person to choose a sandwich. He chooses a sandwich with wheat bread but does not pay attention to what cheese is inside. What is the probability that the cousin chooses a sandwich with cheddar cheese?



1 Determine what probability to find.

Let W represents the set of sandwiches made with wheat bread.

Let C represent the set of sandwiches made with cheddar.

The question asks for $P(C \mid W)$.

2 Calculate $P(C \mid W)$.

The tree diagram shows that

$P(C \cap W) = \frac{3}{8}$ and $P(W) = \frac{3}{4}$.

$$P(C \mid W) = \frac{P(C \cap W)}{P(W)} = \frac{\frac{3}{8}}{\frac{3}{4}} = \frac{3}{8} \cdot \frac{4}{3} = \frac{12}{24} = \frac{1}{2}$$

▶ Given that Antoine's cousin chose a sandwich with wheat bread, the probability that the sandwich had cheddar cheese is $\frac{1}{2}$.

TRY

If Antoine's cousin had chosen a sandwich with rye bread instead, would the probability of choosing a sandwich with cheddar cheese have been the same or different? If different, identify the new probability.

Problem Solving

READ

A total of 90 students (30 students each in grades 10, 11, and 12) at Davis High School were asked if they own smartphones. Of those surveyed, 8 students in 10th grade, 12 students in 11th grade, and 13 students in 12th grade own smartphones. Is owning a smartphone independent of grade level?

PLAN

Create a ______________________ table to show the survey results.

If smartphone owners make up set S and students in particular grades make up sets G_{10}, G_{11} and G_{12}, determine if $P(S \mid G) = P(S)$.

SOLVE

The problem states that the total number of students surveyed in each grade was 30. Write that as the total for each grade-level row. Fill in the rest of the table.

	Smartphone (S)	No Smartphone (N)	Total
10th Grade (G_{10})	8		
11th Grade (G_{11})			
12th Grade (G_{12})			
Total			90

Determine if $P(S \mid G_{10}) = P(S)$.

$P(S \cap G_{10}) = \frac{8}{90} =$ ______________

$P(G_{10}) = \frac{\square}{90} =$ ______________

$P(S \mid G_{10}) = \frac{P(S \cap G_{10})}{P(G_{10})} = \frac{\square}{\square} =$ ______________

$P(S) = \frac{\square}{90} =$ ______________

Does $P(S \mid G_{10}) = P(S)$? ______________

CHECK

Verify that the $P(S \mid G_{11}) \neq P(S)$ and $P(S \mid G_{12}) \neq P(S)$.

$P(S \mid G_{11}) =$ ______________ $P(S \mid G_{12}) =$ ______________

▶ So, owning a smartphone ______________ independent of grade level.

Practice

For questions 1 and 2, a white number cube and a red number cube, each with faces numbered 1 to 6, are tossed at the same time.

1. What is the probability of both cubes landing on 4 if you know that the white cube landed on 4?

   ____

2. What is the probability of both cubes landing on even numbers if you know that the white cube landed on 4? ________________

REMEMBER $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

For questions 3 and 4, a quarter, a nickel, and a penny are all tossed at the same time.

3. What is the probability of all 3 coins landing on heads if you know that the quarter and the nickel landed on heads? ________________

4. What is the probability of all 3 coins landing on heads if all you know is that the quarter landed on heads? ________________

HINT Write the sample space and consider how the conditions change it.

For questions 5–7, a bag contains 3 green marbles and 7 yellow marbles. Parker will draw a marble at random and then draw a second marble without replacing the first marble.

5. Are the events of drawing the first marble and the second marble dependent or independent? Explain.

   ____

   ____

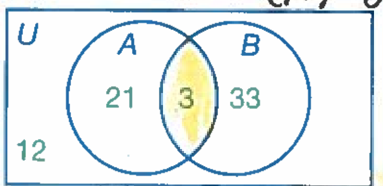
6. Use the general Multiplication Rule to find the probability of drawing a green marble followed by a yellow marble. ________________

7. Find *P*(yellow first and yellow second). ________________

Choose the best answer.

8. The Venn diagram shows the number of possible outcomes in sets *A* and *B* and their intersection. What is $P(B \mid A)$ for this Venn diagram?

$P(A) = 21 + 3$



$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{3}{24}$

A. $\frac{1}{12}$

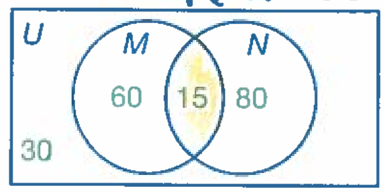
B. $\frac{1}{11}$

C. $\frac{1}{8}$

D. $\frac{1}{7}$

9. The Venn diagram shows the number of possible outcomes in sets *M* and *N* and their intersection. What is $P(N \mid M)$ for this Venn diagram?

$P(M) = 60 + 15 = 75$



A. 0.1875

B. 0.2

C. 0.25

D. 0.5

$P(N|M) = \frac{P(N \cap M)}{P(M)}$

$= \frac{15}{75} = .2$

10. The two-way frequency table below shows the quiz scores of students in the same biology class and whether or not each student studied for the quiz.

	Score $\geq$ 90	Score < 90	
Studied	12	3	15
Did Not Study	1	14	15
	13	17	30

What is the probability that a randomly selected student from the class had a score less than 90, given that the student did not study for the quiz?

A. $\frac{7}{15}$

B. $\frac{17}{30}$

C. $\frac{4}{5}$

D. $\frac{14}{15}$

$P(\text{score} < 90 \mid \text{not study})$

$= \frac{P(\text{score} < 90 \text{ and did not study})}{\text{did not study}} = \frac{14}{15}$

11. The two-way frequency table below shows the number of boys and girls who are working on the school play and whether they are performers or stage-crew members.

	Performer	Stage Crew
Boys	14	18
Girls	21	7
		25

← Boy in stage crew

What is the chance that a randomly selected student working on the school play is a boy, given that the student is a member of the stage crew?

A. 30%

B. 56.25%

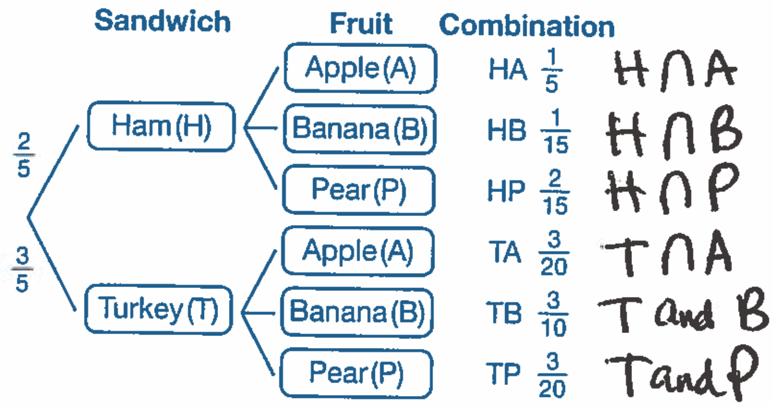
C. 72%

D. 78.26%

$\frac{18}{25} = 72\%$

Use the information and tree diagram below for questions 12 and 13.

There were 50 boxed lunches prepared for a school meeting. Each boxed lunch contains either a ham sandwich or a turkey sandwich and either an apple, a banana, or a pear. Alice is the first to choose a boxed lunch. She chooses a box marked "ham sandwich," but she does not pay attention to which type of fruit is inside.



12. Given that she chose a ham sandwich, what is the probability that she chose a boxed lunch with

an apple? 1/2 $P(\text{apple} \mid \text{Ham}) = \dfrac{P(A \cap H)}{P(H)} = \dfrac{\frac{1}{5}}{\frac{2}{5}}$

with a banana? 1/6 $P(B \mid \text{Ham}) = \dfrac{P(B \text{ and } H)}{P(H)} = \dfrac{\frac{1}{15}}{\frac{2}{5}} =$

with a pear? 1/3 $P(\text{Pear} \mid \text{Ham})$ $\dfrac{\frac{2}{15}}{\frac{2}{5}} = \frac{1}{3}$

13. Compare $P(A \mid T)$ to $P(P \mid T)$.

$P(A \mid T) = \dfrac{P(A \cap T)}{P(T)} \quad \dfrac{\frac{3}{20}}{\frac{3}{5}} = \frac{1}{4}$

$P(\text{Pear} \mid T) = \dfrac{P(P \cap T)}{P(B)} = \frac{1}{4}$

Choose the best answer.

14. Events A and B are dependent events. Which of the following is true?

A. $P(A \mid B) = P(A)$

B. $P(A \mid B) = \dfrac{P(A \cap B)}{P(A)}$

C. $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$ ($A \cap B$)

D. $P(A \text{ and } B) = P(A) \cdot P(B)$

$\dfrac{P(B \mid A)}{1} = \dfrac{P(A \cap B)}{P(A)}$

$P(A \cap B) = P(A) \cdot P(B \mid A)$

15. A bag contains 2 blue, 5 yellow, and 1 black tile. What is the probability of drawing a blue tile followed by a black tile if the first tile drawn is **not** replaced?

A. $\frac{1}{32}$

B. $\frac{1}{28}$

C. $\frac{1}{12}$

D. $\frac{1}{8}$

$\frac{2}{8} \cdot \frac{1}{7}$

$\frac{1}{4} \cdot \frac{1}{7} = \frac{1}{28}$

$P(\text{Blue}, \text{Black})$

$$\frac{\frac{3}{350}}{\frac{75}{350}}$$

Use the two-way frequency table and information below for questions 16–19.

A poll asked voters of different ages if they were for or against a proposal to build a new public library.

Ages	For	Against	No Opinion	Total
18–39	30	42	3	75
40–59	80	55	5	140
60 and over	120	13	2	135
Total	230	110	10	350

16. What is the chance that a voter has no opinion on the proposal, given that the voter is younger than 40 years old?

P(no opin | 18-39) = $\frac{3}{75}$ = $\frac{1}{25}$

17. What is the probability that a voter is for the proposal, given that he or she is 60 or over?

P(FOR | 60 & over) = $\frac{120}{135}$ = $\frac{8}{9}$

18. What is the probability that a voter who is against the proposal is age 40–59?

____

19. Is a voter's opinion about the proposed new library independent of age? How do you know?

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Use the information below for questions 20 and 21.

A total of 60 students in grades 10, 11, and 12 were surveyed and asked whether or not they play on a sports team. Of those surveyed, 20 out of 30 tenth-grade students play on a team, 8 out of 12 eleventh-grade students play on a team, and 12 out of 18 twelfth-grade students play on a team.

20. CREATE Enter the above information in the two-way frequency table. List the joint and marginal frequencies.

	Play on Team(s)	No Team	Total
10th Grade			
11th Grade			
12th Grade			
Total			

21. JUSTIFY Is participation in team sports independent of grade level? Justify your answer.

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